Matrix and Determinant

1. Use the Crammer's rule to discuss the consistency of the following system of equation for different cases of λ:

$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}$$

2. Let $A = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $\alpha \neq 1$.

(a) Prove that for all positive integer n,

$$A^n = \begin{pmatrix} \alpha^n & \dfrac{\beta(\alpha^n - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$$

(b) **(i)** Find BA.

(ii) Use **(a)** , or otherwise, evaluate $(BA)^n$, for $n \in \mathbf{N}$,

3. **(a)** Factorize $F(\alpha, \beta, \gamma) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$.

(b) Show that $\begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = k\, F(x,y,z)\, F(a,b,c)$

where k is a constant to be found, and hence factorize the determinant.

4. If n is the least positive integer such that A^n is a zero matrix, then A is said to be nilpotent of order n.

Given A is a nilpotent of order n.

(a) **(i)** Evaluate $(I-A)\left(I+A+A^2+\ldots+A^{n-1}\right)$ and

$$(I+A)\left\{I-A+A^2-\ldots+(-1)^{n-1}A^{n-1}\right\}$$

(ii) Hence, or otherwise, express $(I-A)^{-1}$ and $(I-A^2)^{-1}$ in terms of A.

(b) Let $A=\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

(i) Evaluate A^2 and A^3.

(ii) Using (a), or otherwise, find $(I-A)^{-1}$ and $(I-A^2)^{-1}$.

5. Let $A=\begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

(a) Find A^3-2A^2-7A+I where I is the identity matrix of order 3 x 3.

(b) Using (a), evaluate $(A-I)(A^2-A-8I)$.

(c) Hence find $(A^2-A-8I)^{-1}$.

6. Find the equation of the image of the curve :

$$5x^2-2\sqrt{3}xy+7y^2-4=0$$

if the **curve** is under the rotation transformation through an angle $\frac{5\pi}{6}$ anti-clockwisely about the origin.

Hint : The formula for rotating anti-clockwisely by an angle θ is

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$